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The Asymptotic Behavior of a Lindelof Function and Its Taylor Coefficients

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1. INTRODUCTION

Let $P_\lambda(z)$ denote the canonical product formed with the sequence of zeros $\{z_n\}$ defined by

$$-z_n = n^{1/\lambda} \qquad (n = 1, 2, 3,...; 0 < \lambda < \infty). \tag{1}$$

This paper is devoted to the study of the asymptotic behavior of both $P_\lambda(z)$ and the coefficients in its power series expansion

$$P_\lambda(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{2}$$

If $0 < \lambda < 1$, P_λ is an "admissible" function in the sense of Hayman [5] and the study of its Taylor's series coefficients is complete [5, p. 69]. But if $1 < \lambda < \infty$ and λ is not an integer, then the results of this paper will show that P_λ is not admissible. Nevertheless, P_λ retains many of the properties of admissible functions and consequently the behavior of its coefficients is described by[1]

THEOREM 5. *If $1 < \lambda < \infty$ and λ is not an integer, then the coefficients a_n of P_λ satisfy the following*:

(I) *We have*

$$a_n = \sqrt{\frac{2}{\pi\lambda n}} \left\{\frac{n\,|\sin \pi\lambda|}{\pi\lambda}\right\}^{-(n+1/2)/\lambda} (2\pi)^{-1/2\lambda}\, e^{n/\lambda + O(n^{q/\lambda})}$$
$$\times \{\cos(H(n)) + o(1)\}, \tag{3}$$

where $H(n)$ is an unbounded strictly decreasing function for all large n.

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[1] We present our main results with Theorem numbers corresponding to their order of proof in the later sections of the paper.

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409/93/2-14

(II) *For every positive* $\eta < \pi/2$, *there exist two unbounded sequences* $\{\nu_k\}$, $\{\mu_k\}$

$$\nu_k < \mu_k < \nu_{k+1} < \mu_{k+1}, \qquad k = 1, 2, \ldots,$$

of positive strictly increasing real numbers such that

(i) *as* $k \to \infty$

$$\nu_k \sim (\lambda/q)k \sim \mu_k \tag{4}$$

and more precisely

$$\mu_k - \nu_k \to (\lambda/q)(1 - (2\eta/\pi)), \qquad \nu_{k+1} - \mu_k \to (\lambda/q)(2\eta/\pi); \tag{5}$$

(ii) *for all n such that*

$$\nu_k \leqslant n \leqslant \mu_k$$

the coefficients a_n *are not zero, and they all have the same sign. Furthermore, the sign of* a_n *changes as we pass to the next interval* $\nu_{k+1} \leqslant n \leqslant \mu_{k+1}$.

We prove part II of Theorem 5 by the method of Edrei [2, p. 224]. For part I, the principle of our proof does not differ from that of Hayman's method [5, p. 70]. It is not, however, a completely trivial extension of [5], because for $\lambda > 1$, the function P_λ has several maxima on the circle $|z| = r$; and the contribution of these points to the size of $|P_\lambda|$ must be assessed. Thus it is necessary to obtain an asymptotic expansion of $P_\lambda(z)$ containing more terms than the well-known Lindelof expansion [8, p. 53]

$$\log P_\lambda(z) = (\pi \csc \pi\lambda)\, z^\lambda (1 + \varepsilon(z)), \tag{6}$$

where

$$-\pi < \arg z < \pi, \qquad q = [\lambda],$$

and $\varepsilon(z)$ tends to 0 uniformly in

$$-\pi + \delta < \arg z < \pi - \delta \qquad (0 < \delta < \pi).$$

Indeed, Lindelof's formula above suggests the values

$$\arg z = k\pi/\lambda, \qquad |k| = 0, 1, \ldots, [\lambda],$$

as the possible arguments of the points on $|z| = r$ where $|P_\lambda(z)|$ assumes an extreme value. But we cannot, by use of this formula, decide which among these values gives the point where $\max |P_\lambda(z)|$ is attained. Nor can we get a useful estimate for the ratio of the values of $|P_\lambda(z)|$ at two of these points.

The "complete" asymptotic expansion of $\log P_\lambda(z)$ was given by Barnes in a very long paper [1] in which a certain summability theory of series is developed and then combined with the Euler–Maclaurin formula to obtain the desired expansion of $\log P_\lambda(z)$. Barnes's paper contains no real proofs and his formulas are a result of formal manipulations of ordinary and asymptotic series; the validity of those manipulations does not appear to be easily demonstrable.

A study of P_λ undertaken by Ford [4] resulted in a formula which in Ford's words [4, p. 56] "is not altogether consistent with that of Barnes." Ford did not assert that the formula of Barnes was incorrect.

Much earlier, Hardy [6] had used the formula of Barnes in studying the behavior of $P_\lambda(z)$ on the negative x-axis, a case not covered by Barnes's work. Hardy's appeal to the work of Barnes, in [6] and the subsequent correction [7], may be regarded as strong support for the results of Barnes. But Ford also shows that his formulas agree with those of Hardy near the negative x-axis [4, p. 56].

To sum up, despite the existence of "complete" asymptotic formulas for $P_\lambda(z)$, it appeared that there was some uncertainty surrounding these formulas. Rather than try to check or, if necessary, correct the proofs in [1, 4, 6], we thought it more constructive to start afresh and devise simple and short proofs of an asymptotic formula for $P_\lambda(z)$.

In order to state our formulas we introduce the function $\xi(t)$ defined by

$$\xi(t) = \int_0^t ([x] - x + \tfrac{1}{2})\, dx \qquad (t \geqslant 0), \tag{7}$$

where $[x]$ is the greatest integer $\leqslant x$.

For a positive real number δ $(0 < \delta < \pi)$, we let $D = D_\delta$ be the region in the complex plane defined by

$$D = \{z = re^{i\theta}: r > 0, -\pi + \delta < \theta < \pi - \delta\}. \tag{8}$$

The zeta function of Riemann will, as usual, be denoted by $\zeta(z)$. A sum whose upper index is less than its lower index is to be interpreted as an empty sum.

With the above notation, our formula for $P_\lambda(z)$ is given in

THEOREM 1. *Assume that λ is nonintegral. If $q = [\lambda]$ and $z \in D$, then*

$$\log P_\lambda(z) = (\pi \csc \pi\lambda)\, z^\lambda + \sum_{m=1}^{q} \frac{(-1)^m}{m} \zeta(m/\lambda)\, z^m - \frac{1}{2} \log z - \frac{1}{2\lambda} \log 2\pi + \mathscr{E}(z), \tag{9}$$

where all the logarithms have their principal values,

$$\mathscr{E}(z) = \int_0^\infty \frac{\{(\lambda^{-1} - 1)z - t\}\, \xi(t^\lambda)}{(z+t)^2\, t^\lambda}\, dt, \tag{10}$$

and $\mathscr{E}(z)$ *tends to zero as* z *tends to* ∞ *uniformly in* D.

Combining Theorem 1 with a result of Williamson [11], we obtain very useful information about the local and global maxima of $|P_\lambda(z)|$ on $|z| = r$.

Assume that λ (>1) is nonintegral. Let $q = [\lambda]$ and define an index set J by

$$\begin{aligned} J &= \{k: -\tfrac{1}{2}q \leqslant k \leqslant \tfrac{1}{2}q\}, && \text{if} \quad q \text{ is even}, \\ &= \{k: -\tfrac{1}{2}(q+1) \leqslant k \leqslant \tfrac{1}{2}(q-1)\}, && \text{if} \quad q \text{ is odd}. \end{aligned} \tag{11}$$

Next define the arugments $\theta_k \in (-\pi, \pi)$ as follows:

$$\begin{aligned} \theta_k &= 2k\pi/\lambda, && k \in J, \quad q \text{ even}, \\ &= (2k+1)\pi/\lambda, && k \in J, \quad q \text{ odd}, \end{aligned} \tag{12}$$

and let

$$\theta_\omega = q\pi/\lambda$$

denote the "last" argument.

For each nonnegative $k \in J$ we define the constant α_k by

$$-\alpha_k = ((-1)^q/q)\, \zeta(q/\lambda)\{\cos q\theta_k - \cos q\theta_{k+1}\}. \tag{13}$$

THEOREM 4. *Let* $\delta = \frac{1}{4}\pi(1 - q/\lambda)$ *and retain the notations of the previous paragraph. Then there exists a positive real number* r_0 *such that, for all* $r \geqslant r_0$, $|P_\lambda(re^{i\theta})|$ *has exactly* $q + 1$ *points of local maxima in the interval* $[-\pi, \pi]$, *one occurring in each of the intervals*

$$|\theta - \theta_k| \leqslant \delta \qquad (k \in J).$$

Furthermore, if we denote by $\beta_k = \beta_k(r)$ *the maximum point occurring in* $|\theta - \theta_k| \leqslant \delta$, *then*

$$\beta_k \to \theta_k \qquad \text{as} \quad r \to \infty, \quad (k \in J); \tag{14}$$

$$|P_\lambda(re^{i\beta_k})|/|P_\lambda(re^{i\beta_{k+1}})| = O(\exp(-\tfrac{1}{2}\alpha_k r^q)) \tag{15}$$

as $r \to \infty$ *uniformly for all nonnegative* $k \in J$ *such that* $(k+1) \in J$.

If $M(r) = \max |P_\lambda(re^{i\theta})|$ *in* $-\pi \leqslant \theta \leqslant \pi$, *then, for all large* r, *$M(r)$ is attained at exactly two points in* $[-\pi, \pi]$, *namely,* $\pm\beta_\omega$. (16)

Since $\theta_\omega = q\pi/\lambda \geqslant \frac{1}{2}\pi$ as soon as $q \geqslant 1$, we see from (14) that the functions P_λ achieve their maximum modulus in the left half plane when $q \geqslant 1$, thus providing a negative answer to a question of Williamson [11, p. 511].

Another consequence of (15) is

COROLLARY 1. *If λ (>1) is nonintegral, then the coefficients a_n defined by* (2) *change their sign infinitely often.*

Only little modification of our proof of Theorem 1 is needed to handle the case when λ is an integer and we obtain

THEOREM 2. *Assume that $\lambda = q$ is a positive integer and let c be Euler's constant. If $z \in D$, then*

$$\log P_\lambda(z) = (-z)^\lambda \log z + \frac{(c-1)}{\lambda}(-z)^\lambda + \sum_{m=1}^{\lambda-1} \frac{(-1)^m}{m} \zeta(m/\lambda)\, z^m$$

$$-\tfrac{1}{2}\log z - (1/2\lambda)\log 2\pi + \mathscr{E}(z), \tag{17}$$

where all the logarithms and powers have their principal values and $\mathscr{E}(z)$ is the function defined in (10); *in particular $\mathscr{E}(z)$ tends to* 0 *as z tends to ∞ uniformly in D.*

Using Theorems 1 and 2 and a corollary of a lemma of Petrenko [3], we are able to study the behavior of P_λ on the negative x-axis. Retaining the notations of the previous paragraphs our result is

THEOREM 3. *Assume that $\lambda > \frac{1}{2}$ and put $\gamma = 2\lambda$. It is then possible to find a positive real number δ such that if*

$$X(t, z) = t^\gamma z^\gamma/(t^\gamma + z^\gamma)^2 \qquad \text{and} \qquad |\arg z| \leqslant \delta,$$

then, with all logarithms and powers having their principal values, we have

(i) *If λ is nonintegral*

$$P_\lambda(-z) = 2(2\pi)^{-(1/2\lambda)} z^{-1/2} \sin \pi z^\lambda \cdot$$

$$\times \exp\left\{\pi \cot \pi\lambda z^\lambda + \sum_{m=1}^{q} \frac{\zeta(m/\lambda)}{m} z^m - \mathscr{E}_1(z)\right\}. \tag{18}$$

(ii) *If λ is a positive integer q*

$$P_\lambda(-z) = 2(2\pi)^{-(1/2\lambda)} z^{-1/2} \sin \pi z^\lambda \exp\left\{z^q \log z + \frac{(c-1)}{q} z^q\right.$$

$$\left. + \sum_{m=1}^{q-1} \frac{\zeta(m/\lambda)}{m} z^m - \mathscr{E}_1(z)\right\}. \tag{19}$$

(iii) *In both* (18) *and* (19), *the term* $\mathscr{E}_1(z)$ *is given by*

$$\mathscr{E}_1(z) = \frac{\gamma^2}{2\pi} \int_0^\infty X(t, z) \left\{ \int_{-\pi+\pi/\gamma}^{\pi-\pi/\gamma} \operatorname{Re} \mathscr{E}(te^{i\theta})\, d\theta \right\} dt/t, \tag{20}$$

where $\mathscr{E}(z)$ *is defined by* (10). *Furthermore,* $\mathscr{E}_1(z)$ *tends to* 0 *as* z *tends to* ∞ *uniformly in* $|\arg z| < \delta$.

Formula (18) was obtained by Hardy in [6] and later corrected in [7], with a different form for the error term. Formula (19) appears to be new.

If $0 < \lambda < \frac{1}{2}$, the shortest way to get the behavior of $P_\lambda(z)$ on the negative x-axis is to follow the idea of Hardy [6], namely, to write

$$P_\lambda(-x) = P_{2\lambda}(\sqrt{x})\, P_{2\lambda}(-\sqrt{x})$$

if, say, $\frac{1}{2} > \lambda > \frac{1}{4}$ and then use (18).

For functions of order 0, one may consider the infinite product

$$\prod_{n=1}^{\infty} (1 + z/e^n)$$

or, more generally, the product

$$\prod_{n=1}^{\infty} (1 + z/q^n),$$

where $q > 1$.

The methods developed in this paper may be applied to obtain analogs of Theorems 2 and 3 for these functions. Since, for the kind of problems considered here, such functions are covered by Hayman's results [5], we omit such results. Finally, we remark that our methods may also be used to obtain asymptotic formulas for canonical products having only real negative zeros situated at

$$-n^{1/\lambda}(\log n)^a(\operatorname{loglog} n)^b \cdots \qquad (a, b, \ldots, \text{ nonnegative integers}).$$

2. The Asymptotic Formula; Case of Nonintegral Orders

Let

$$P_\lambda(z) = \prod_{n=1}^{\infty} E(-z/n^{1/\lambda}, q), \tag{2.1}$$

where λ is a positive noninteger, $q = [\lambda]$, and $E(x, q)$ is the Weierstrass primary factor of genus q.

As usual, we denote by $n(t)$ the number of zeros of P_λ in $|z| \leqslant t$. Thus

$$n(t) = [t^\lambda] \qquad (t \geqslant 0).$$

We introduce the function $\xi(t)$ defined by

$$\xi(t) = \int_0^t ([x] - x + \tfrac{1}{2})\, dx \qquad (t \geqslant 0), \tag{2.2}$$

and note the following two inequalities satisfied by ξ:

$$0 \leqslant \xi(t) \leqslant \tfrac{1}{2}t, \qquad |\xi(t)| \leqslant \tfrac{1}{8} \qquad (t \geqslant 0). \tag{2.3}$$

For a positive real number δ $(0 < \delta < \pi)$, we let $D = D_\delta$ be the region in the complex plane defined by

$$D = \{z = re^{i\theta} : r > 0, -\pi + \delta < \theta < \pi - \delta\}. \tag{2.4}$$

Proof of Theorem 1. Denote by $\{c_m(r)\}$ the Fourier coefficients of $\log|P_\lambda(re^{i\theta})|$; then $c_m(r)$ may be expressed in terms of $n(r)$ as follows [9, p. 380]:

$$2(-1)^m c_m(r) = r^m \int_1^r t^{-m-1} n(t)\, dt + r^{-m} \int_0^r t^{m-1} n(t)\, dt \qquad (q \geqslant 1, 1 \leqslant m \leqslant q),$$

$$-2(-1)^m c_m(r) = r^m \int_r^\infty t^{-m-1} n(t)\, dt - r^{-m} \int_0^r t^{m-1} n(t)\, dt \qquad (m \geqslant q+1), \tag{2.5}$$

$$c_m(r) = c_{-m}(r) \qquad \text{if} \quad m < 0,$$

$$c_0(r) = N(r) = \int_0^r t^{-1} n(t)\, dt.$$

Starting from the equality

$$\log|P_\lambda(re^{i\theta})| = c_0(r) + 2 \sum_{m=1}^{\infty} c_m(r) \cos(m\theta) \qquad (-\pi < \theta < \pi), \tag{2.6}$$

we first derive formula (9) for $\log|P_\lambda(z)|$ with both the logarithmic term and the constant term combined with $\mathscr{E}(z)$. We pass from this formula to one involving $\log P_\lambda(z)$ and then "extract" the logarithmic term and the constant term. In the process we shall need two formulas connecting our function $\xi(t)$ with $\zeta(t)$, the zeta function of Riemann. The first formula

$$\zeta(s) = \frac{s}{s-1} + s \int_1^\infty ([x] - x)\, x^{-s-1}\, dx \qquad (s > 0, s \neq 1), \tag{2.7}$$

is well known. From it we easily obtain

$$\zeta(s)-\frac{1}{s-1}-\frac{1}{2}=s\int_1^{\infty} x^{-s-1}\left([x]-x+\frac{1}{2}\right)dx \qquad (s>0). \quad (2.8)$$

Now in (2.8) we integrate once by parts, divide by s and let $s\to 0$; we arrive at our second formula

$$\int_1^{\infty} t^{-2}\xi(t)\,dt=\zeta'(0)+1=-\tfrac{1}{2}\log 2\pi+1. \quad (2.9)$$

We now proceed to find a formula for $\log|P_\lambda(re^{i\theta})|$. Setting

$$Y(t)=n(t)-t^{\lambda} \qquad (0\leqslant t<\infty), \quad (2.10)$$

we obtain directly from (2.5)

$$2(-1)^m c_m(r)=\frac{2\lambda r^{\lambda}}{\lambda^2-m^2}+\frac{r^m}{m-\lambda}+r^m\int_1^r t^{-m-1}Y(t)\,dt$$
$$+r^{-m}\int_0^r t^{m-1}Y(t)\,dt \quad (2.11a)$$

for $q\geqslant 1$ and $1\leqslant m\leqslant q$; for $m\geqslant q+1$ we have

$$-2(-1)^m c_m(r)=\frac{2\lambda r^{\lambda}}{m^2-\lambda^2}+r^m\int_r^{\infty} t^{-m-1}Y(t)\,dt$$
$$-r^{-m}\int_0^r t^{m-1}Y(t)\,dt. \quad (2.11b)$$

Recalling that $n(t)=[t^{\lambda}]$, a change of variable together with (2.7) gives

$$\int_1^{\infty} t^{-m-1}Y(t)\,dt=\frac{1}{m}\zeta(m/\lambda)+\frac{1}{\lambda-m} \qquad (m/\lambda>0, m\neq\lambda), \quad (2.12)$$

from which it follows that, for $q\geqslant 1$ and $1\leqslant m\leqslant q$,

$$\frac{r^m}{m-\lambda}+r^m\int_1^r t^{-m-1}Y(t)\,dt=\frac{r^m}{m}\zeta(m/\lambda)-r^m\int_r^{\infty} t^{-m-1}Y(t)\,dt. \quad (2.13)$$

Substituting (2.11) in (2.6), using (2.13), and recalling the identity

$$\pi\lambda\csc\pi\lambda\cos\lambda\theta=1+2\sum_{m=1}^{\infty}\frac{(-1)^m\lambda^2}{\lambda^2-m^2}\cos(m\theta),$$

we obtain

$$\begin{aligned}\log|P_\lambda(re^{i\theta})| = c_0(r) &+ (1/\lambda)(\pi\lambda \csc \pi\lambda \cos \lambda\theta - 1)\, r^\lambda \\ &+ \sum_{m=1}^{q} \frac{(-1)^m}{m} \zeta(m/\lambda)\, r^m \cos(m\theta) \\ &+ \sum_{m=1}^{\infty} (-1)^m \left\{ -r^m \int_r^\infty t^{-m-1} Y(t)\, dt \right. \\ &\left. + r^{-m} \int_0^r t^{m-1} Y(t)\, dt \right\} \cos(m\theta),\end{aligned} \tag{2.14}$$

where

$$r > 0, \qquad -\pi < \theta < \pi, \qquad q < \lambda < q+1, \qquad q \geqslant 0,$$

and the first sum in (2.14) is to be omitted if $q = 0$.

Our next step is to sum the infinite series in (2.14). Since there is no absolute convergence, we cannot appeal to the usual theorems for interchanging summation and integration. We therefore proceed as follows:

Fix $\theta \in (-\pi, \pi)$ and consider the series

$$\sum_{m=1}^{\infty} (-1)^m \left(\frac{t}{r}\right)^m (\cos m\theta) \frac{Y(t)}{t} x^m, \tag{2.15}$$

where

$$r > 0, \qquad 0 \leqslant t \leqslant r, \qquad 0 < x < 1.$$

Since $Y(t)$ is bounded for all $t \geqslant 0$, this series is, for each fixed $x \in (0, 1)$, uniformly convergent for all $t \in [0, r]$. The sum is easily computed and term by term integration with respect to t then gives

$$\sum_{m=1}^{\infty} \left\{ (-1)^m r^{-m} (\cos m\theta) \int_0^r t^{m-1} Y(t)\, dt \right\} x^m = \int_0^r \frac{Y(t)}{t} \operatorname{Re}\left(\frac{-xt}{z+xt}\right) dt, \tag{2.16}$$

where $z = re^{i\theta}$, $-\pi < \theta < \pi$, and $0 < x < 1$. If we denote the series in (2.16) by $\sum a_m x^m$, then (2.16) gives

$$\lim_{x\to 1} \sum_{m=1}^{\infty} a_m x^m = \lim_{x\to 1} \int_0^r \frac{Y(t)}{t} \operatorname{Re}\left(\frac{-xt}{z+xt}\right) dt = \operatorname{Re} \int_0^r \frac{-Y(t)}{z+t}\, dt. \tag{2.17}$$

The boundedness of $Y(t)$ and the definition of a_m give

$$|a_m| = \left| \left\{ (-1)^m r^{-m} \int_0^r t^{m-1} Y(t)\, dt \right\} \cos m\theta \right| \leqslant m^{-1} \qquad (m \geqslant 1). \tag{2.18}$$

Now (2.18), (2.17), and Littlewood's theorem [10, p. 233] imply

$$\sum_{m=1}^{\infty} \left\{ (-1)^m r^{-m} \int_0^r t^{m-1} Y(t)\, dt \right\} \cos m\theta = \sum_{m=1}^{\infty} a_m = \operatorname{Re} \int_0^r \frac{-Y(t)}{z+t}\, dt, \tag{2.19}$$

where $z = re^{i\theta}$ and $-\pi < \theta < \pi$.

Similarly, we show that

$$\sum_{m=1}^{\infty} \left\{ (-1)^m r^m \int_r^{\infty} t^{-m-1} Y(t)\, dt \right\} \cos m\theta = \int_r^{\infty} \frac{Y(t)}{t} \operatorname{Re} \left(\frac{-z}{z+t} \right) dt$$
$$(z = re^{i\theta}, -\pi < \theta < \pi). \tag{2.20}$$

An integration by parts gives

$$\int_r^{\infty} \frac{Y(t)}{t} \operatorname{Re} \left(\frac{z}{z+t} \right) dt - \int_0^r \frac{Y(t)}{t} \operatorname{Re} \left(\frac{t}{z+t} \right) dt$$
$$= -X(r) + \operatorname{Re} \int_0^{\infty} \frac{zX(t)}{(z+t)^2}\, dt, \tag{2.21}$$

where

$$X(r) = \int_0^r t^{-1} Y(t)\, dt \qquad (r \geqslant 0).$$

In terms of $X(r)$, the coefficient $c_0(r)$ takes the form

$$c_0(r) = \int_0^r t^{-1} n(t)\, dt = X(r) + \frac{r^{\lambda}}{\lambda}. \tag{2.22}$$

Substituting from (2.19), (2.20), and (2.22) in (2.14) and then using (2.21), we arrive at the formula

$$\log |P_{\lambda}(re^{i\theta})| = (\pi\lambda \csc \pi\lambda \cos \lambda\theta)\, r^{\lambda} + \sum_{m=1}^{q} \frac{(-1)^m}{m} \zeta(m/\lambda)\, r^m \cos m\theta$$
$$+ \operatorname{Re} \int_0^{\infty} \frac{zX(t)}{(z+t)^2}\, dt, \tag{2.23}$$

valid under the same conditions as (2.14).

Since $P_\lambda(r) > 0$, we obtain as a special case of (2.23)

$$\log P_\lambda(r) = (\pi \csc \pi\lambda)\, r^\lambda + \sum_{m=1}^{q} \frac{(-1)^m}{m} \zeta(m/\lambda)\, r^m + \int_0^\infty \frac{rX(t)}{(r+t)^2}\, dt \qquad (r \geqslant 0). \tag{2.24}$$

We now extend (2.24) to the region D.

Fix $\delta \in (0, \pi)$ and let $D = D_\delta$ be the region defined in (2.4). The integral

$$\int_0^\infty \frac{zX(t)}{(z+t)^2}\, dt \tag{2.25}$$

is uniformly convergent on compact subsets of D. To see this, divide the integral into two integrals over $[0, 1]$ and $[1, \infty)$ and, on compact subsets of D, compare the two parts with

$$\csc^2(\delta/2) \int_0^1 t^{\lambda-1}\, dt, \qquad \csc^2(\delta/2) \int_1^\infty t^{-2} \log t\, dt$$

which are convergent and independent of z.

The continuity of the integrand in (2.25) being obvious, we conclude that the integral in (2.25) is analytic in D.

In the plane cut along the negative x-axis, define the two functions z^λ and $\log P_\lambda(z)$ by

$$z^\lambda = \exp\{\lambda\,(\log r + i\theta)\}, \qquad \log P_\lambda(z) = \int_{[0,z]} \frac{P'_\lambda(\zeta)}{P_\lambda(\zeta)}\, d\zeta,$$

where

$$z = re^{i\theta}, \quad -\pi < \theta < \pi, \quad r > 0, \quad [0, z] = \text{line segment from 0 to } z.$$

Then z^λ and $\log P_\lambda(z)$ are analytic and real for real positive z. Thus (2.24) expresses the equality on the positive axis of two functions analytic in D, hence

$$\log P_\lambda(z) = (\pi \csc \pi\lambda)\, z^\lambda + \sum_{m=1}^{q} \frac{(-1)^m}{m} \zeta(m/\lambda)\, z^m + \int_0^\infty \frac{zX(t)}{(z+t)^2}\, dt \tag{2.26}$$

for all z satisfying $-\pi < \arg z < \pi$.

It remains for us to extract a logarithmic term and a constant term from the integral in (2.26) and it is here that we use (2.2) and (2.9).

From the definition of $X(t)$ and a change of variable, we get

$$X(t) = \int_0^t x^{-1}([x^\lambda] - x^\lambda)\, dx \tag{2.27}$$

$$= -\frac{1}{\lambda} - \frac{1}{2}\log t + \frac{\xi(t)}{\lambda t^\lambda} + \int_1^{t^\lambda} \frac{\xi(x)}{\lambda x^2}\, dx, \qquad \text{if} \quad t > 0,$$

$$= 0, \qquad \text{if} \quad t = 0.$$

Now (2.27) and (2.9) give

$$X(t) = -\frac{1}{2}\log t - \frac{1}{2\lambda}\log 2\pi + \frac{\xi(t)}{\lambda t^\lambda} - \int_{t^\lambda}^\infty \frac{\xi(x)}{\lambda x^2}\, dx, \qquad \text{if} \quad t > 0,$$

$$= 0; \qquad \text{if} \quad t = 0. \tag{2.28}$$

Substituting from (2.28) and doing one integration by parts gives

$$\int_0^\infty \frac{zX(t)}{(z+t)^2}\, dt = -\frac{1}{2}\log z - \frac{1}{2\lambda}\log 2\pi + \int_0^\infty \frac{\xi(t^\lambda)\{(\lambda^{-1} - 1)z - t\}}{t^\lambda (z+t)^2}\, dt. \tag{2.29}$$

Combining (2.29) and (2.26) we obtain formulas (9) and (10) of Theorem 1.

To prove the last part of Theorem 1, let

$$a = \min(\lambda, 1/\lambda) \qquad (\lambda > 0, \text{ possibly an integer})$$

and write $\mathscr{E}(z) = I_1 + I_2$, where I_1 is an integral over $[0, r^a]$, I_2 is an integral over $[r^a, \infty]$, and $z \in D$.

Using the first inequality of (2.3) in I_1 and the second in I_2 we obtain

$$|I_1| \leqslant \tfrac{1}{2}(\lambda^{-1} + 1)\csc^2(\delta/2)\log((r + r^a)/r) \leqslant K r^{a-1}$$

and

$$|I_2| \leqslant ((\lambda^{-1} + 1)/8\lambda)\csc^2(\delta/2)\, r^{-a\lambda}.$$

If $a \neq 1$, then $a < 1$ and both I_1 and I_2 tend to zero as $|z| \to \infty$ uniformly in D. If $a = 1$, then $\mathscr{E}(z)$ takes the simple form

$$\mathscr{E}(z) = -\int_0^\infty \frac{\xi(t)}{(z+t)^2}\, dt,$$

which is easily seen to tend to 0 as $|z| \to \infty$, uniformly in D. This completes the proof of Theorem 1.

3. The Asymptotic Formula; Case of Integral Orders

Throughout this section we assume that the order λ is a positive integer q and use the letters λ and q interchangeably. We retain all the notations of Section 2 and in addition we let c denote Euler's constant.

Proof of Theorem 2. Let $\{c_m(r)\}$ be the Fourier coefficients of $\log|P_\lambda(re^{i\theta})|$. If $Y(t)$ is the function defined in (2.10), then, for $m \neq q$, $c_m(r)$ is given by (2.11); but for $m = q$ we have instead

$$\begin{aligned} 2(-1)^q c_q(r) &= r^q \log r + \frac{r^q}{2q} + r^q \int_1^r t^{-q-1} Y(t)\,dt + r^{-q} \int_0^r t^{q-1} Y(t)\,dt \\ &= r^q \log r + \frac{r^q}{2q} + (c-1)\frac{r^q}{q} \\ &\quad - r^q \int_r^\infty t^{-q-1} Y(t)\,dt + r^{-q} \int_0^r t^{q-1} Y(t)\,dt. \end{aligned} \tag{3.1}$$

Formula (2.13) continues to hold when $q \geqslant 2$ and $1 \leqslant m < q$, but the identity following it must be replaced by the identity

$$\sum_{|m| \neq q} \frac{(-1)^m \lambda^2}{\lambda^2 - m^2} e^{im\theta} = \frac{(-1)^q}{2q} (-2\theta q^2 \sin q\theta - q \cos q\theta) \qquad (\lambda = q), \tag{3.2}$$

which is easily obtained from the corresponding identity in the nonintegral case, or by direct computation of the Fourier series of the function on the right-hand side of (3.2).

Proceeding as in Section 2, we first obtain the formula

$$\begin{aligned} \log|P_\lambda(re^{i\theta})| &= c_0(r) - \frac{r^q}{q} + \operatorname{Re}\{(-1)^q z^q \log z\} \\ &\quad + \frac{(-1)^q}{q}(c-1)\, r^q \cos q\theta \\ &\quad + \sum_{m=1}^{q-1} \frac{(-1)^m}{m} \zeta(m/\lambda)\, r^m \cos m\theta + \operatorname{Re} \int_0^\infty \frac{zX(t)}{(z+t)^2}\,dt, \end{aligned} \tag{3.3}$$

where $\log z = \log r + i\theta$ for $z = re^{i\theta}$ and $-\pi < \theta < \pi$, and the sum appearing in (3.3) is to be omitted if $q = 1$. From this point on the steps in the proof are identical to those of Section 2 occurring after (2.23) and we arrive at the formula for $\log P_\lambda(z)$ stated in Theorem 2.

Remark. For $\lambda = 1$, the formula in Theorem 2 coincides with Stirling's formula as given in [10, p. 151]. The function $\xi(t)$ can be written

$$\xi(t) = \sum_{n=1}^{\infty} \frac{(1 - \cos 2n\pi t)}{2\pi^2 n^2}.$$

The series may be inserted in the formula for $\mathscr{E}(z)$ and integrated term by term as suggested in [10, p. 151, ex. (iii)] to obtain further terms in the asymptotic expansion. The same procedure may be followed when $\lambda \neq 1$, but the integrals that arise in this case are much more complicated.

4. Behavior on the Negative x-axis

In this section we obtain the proof of Theorem 3. The first step consists in expressing $\log |P_\lambda(-x)|$, where $x > 0$, in terms of integrals of $\log |P_\lambda(re^{i\theta})|$ over subintervals of $[-\pi, \pi]$. This we do using the following lemma whose proof follows easily from Petrenko's formula [3, p. 95] and straightforward estimates:

Lemma 1. *Let f be an entire function of finite order λ. Assume $\gamma = \max(2\lambda, 1)$ and let*

$$X(t, u) = X(t, u; \gamma) = t^\gamma u^\gamma/(t^\gamma + u^\gamma)^2, \tag{4.1}$$

$$H(z, \zeta) = H(z, \zeta; \gamma) = \log((|z|^\gamma + |\zeta|^\gamma)/|z^\gamma - \zeta^\gamma|), \tag{4.2}$$

where ζ may be complex.

Then if $u > 0$, $\varphi \in [-\pi, \pi]$, and $f(ue^{i\varphi}) \neq 0$, we have

$$\log |f(ue^{i\varphi})| = \frac{\gamma^2}{2\pi} \int_0^\infty X(t, u) \left\{ \int_{-\pi/\gamma}^{\pi/\gamma} \log |f(te^{i(w+\varphi)})| \, dw \right\} \frac{dt}{t} - \sum_{|a_n| < \infty} H(a_n e^{-i\varphi}, u), \tag{4.3}$$

where $\{a_n\}$ is the sequence of zeros of f lying in the angle

$$0 < |z| < \infty, \qquad \varphi - \pi/\gamma \leqslant \arg z \leqslant \varphi + \pi/\gamma.$$

Remark. For (4.3) to hold true, the choice $\gamma > \max(\lambda, 1)$ suffices. Furthermore, if $\gamma = 1 > \lambda$, formula (4.3) still holds true [3, p. 96] and does not require that $f(e^{i\pi}r)$ and $f(e^{-i\pi}r)$ be equal.

Our next lemma shows that the behavior of P_λ on the negative x-axis is related to its behavior in the right half plane, in a manner very much like the behavior of the gamma function on the negative x-axis.

LEMMA 2. *If $\lambda \geqslant \frac{1}{2}$, $\gamma = 2\lambda$, and $0 < u \neq n^{1/\lambda}$, $n = 1, 2, 3,\ldots$, then we have*

$$\log |P_\lambda(-u)| = \log |(\sin \pi u^\lambda)/\pi u^\lambda| - \frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi+\pi/\gamma}^{\pi-\pi/\gamma} \log |P_\lambda(te^{i\theta})|\, d\theta \right\} \frac{dt}{t}. \quad (4.4)$$

Proof. Let $g(z) = (\sin \pi z)/\pi z$. Then g is entire and of order 1 and its zeros are located at $n = \pm 1, \pm 2, \pm 3, \pm \cdots$. Apply Lemma 1 with $\gamma = 2$ to $g(z)$ once with $\varphi = 0$ and another time with $\varphi = \pi$. Add the two results, use $g(-u) = g(u)$ and

$$\int_{-\pi/2}^{\pi/2} \log |g(te^{i\theta})|\, d\theta + \int_{-\pi/2}^{\pi/2} \log |g(te^{i(\theta+\pi)})|\, d\theta = 2\pi N(t, 1/g),$$

which is true by Jensen's formula [10, p. 125] since $g(0) = 1$. The result is

$$\log |g(u)| = 2 \int_0^\infty X(t, u; 2)\, N(t, 1/g) \frac{dt}{t} - \sum_{n=1}^\infty \log \frac{n^2 + u^2}{|n^2 - u^2|}, \quad (4.5)$$

provided that $u > 0$ and u is not a positive integer. It follows that

$$\log |g(u^\lambda)| = 2 \int_0^\infty X(t, u^\lambda; 2)\, N(t, 1/g) \frac{dt}{t} - \sum_{n=1}^\infty \log \frac{n^2 + u^{2\lambda}}{|n^2 - u^{2\lambda}|}, \quad (4.6)$$

provided that $0 < u \neq n^{1/\lambda}$, $n = 1, 2, 3,\ldots$.

Next apply Lemma 1 to the function $P_\lambda(z)$ of order λ ($\geqslant \frac{1}{2}$) with $\gamma = 2\lambda$ and $\varphi = \pi$ to obtain

$$\log |P_\lambda(-u)| = \frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi/\gamma}^{\pi/\gamma} \log |P_\lambda(te^{i(\theta+\pi)})|\, d\theta \right\} \frac{dt}{t} - \sum_{n=1}^\infty \log \frac{n^2 + u^{2\lambda}}{|n^2 - u^{2\lambda}|}, \quad (4.7)$$

where $u > 0$ and u^λ is not an integer.

It is our intention to subtract (4.7) from (4.6) to eliminate the sum involving the zeros; but before doing so, note that a simple change of variable implies

$$N(t, 1/P_\lambda) = \int_0^t [s^\lambda]\, s^{-1}\, ds = \frac{1}{2\lambda} \int_0^{t^\lambda} [s] s^{-1}\, ds = \frac{1}{2\lambda} N(t^\lambda, 1/g); \quad (4.8)$$

and (4.8) and another change of variable give

$$\int_0^\infty X(t, u^\lambda; 2)\, N(t, 1/g)\, dt/t = \lambda \int_0^\infty X(t^\lambda, u^\lambda; 2)\, N(t^\lambda, 1/g)\, dt/t$$

$$= 2\lambda^2 \int_0^\infty X(t, u; \gamma)\, N(t, 1/P_\lambda)\, dt/t. \quad (4.9)$$

Using (4.9) in (4.6), subtracting the result from (4.7) and then using Jensen's formula we arrive at (4.4). This completes the proof of Lemma 2.

Proof of Theorem 3. Assume that $\lambda > \frac{1}{2}$ and u is a positive real number such that u^λ is not an integer. Then $\log |P_\lambda(-u)|$ is given by formula (4.4) of Lemma 2. The integral appearing in this formula involves the values of $\log |P_\lambda(re^{i\theta})|$ in the region $\{r > 0, |\theta| \leqslant \pi - \pi/\gamma\}$. Such a region is of the type in which the expansions obtained in Theorems 1 and 2 are valid. We can therefore substitute these expansions into (4.4) and evaluate the various integrals that arise to obtain the expansion of $\log |P_\lambda(-x)|$ for $x > 0$.

When one of the formulas (9) or (17) is used in (4.4), there arise integrals of the form

$$\frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma)\, \psi(t)\, dt/t, \quad (4.10)$$

where $\gamma \geqslant 2\lambda$ (>1) and $\psi(t)$ is one of the functions

$$t^\lambda, \qquad t^m, \qquad t^m \log t, \qquad \log t \qquad (1 \leqslant m \leqslant q = [\lambda] > 0).$$

To evaluate such integrals, we apply Lemma 1 to the function $\exp(\psi(z))$ with $\varphi = 0$ and $\gamma = 2\lambda$, where λ is the order of $\exp \psi(z)$.

Consider first the case

$$\psi(z) = z^\lambda \qquad (\lambda > \tfrac{1}{2}).$$

Let $\gamma = 2\lambda$ and put $z = te^{i\theta}$. Then Lemma 1 applied to $\exp \psi(z)$ gives

$$u^\lambda = \log |e^{u^\lambda}| = \frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi/\gamma}^{\pi/\gamma} \log |e^{z^\lambda}|\, d\theta \right\} dt/t$$

$$= (4\lambda/\pi) \int_0^\infty X(t, u; 2\lambda) t^\lambda\, dt/t. \quad (4.11)$$

From (4.11) follows immediately

$$\frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi+\pi/\gamma}^{\pi-\pi/\gamma} t^\lambda \cos \lambda\theta\, d\theta \right\} dt/t = -u^\lambda \cos \pi\lambda \qquad (\gamma = 2\lambda). \quad (4.12)$$

Consider next the case

$$\psi(z) = z^q \log z,$$

where q is a positive integer and the logarithm has its principal value.

Let $\gamma = 2q$ and put $z = te^{i\theta}$. Then Lemma 1 applied to $\exp \psi(z)$ gives

$$\begin{aligned} u^q \log u &= \frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi/\gamma}^{\pi/\gamma} (t^q \log t \cos q\theta - t^q \theta \sin q\theta) \right\} dt/t \\ &= \frac{\gamma^2}{q\pi} \int_0^\infty X(t, u; \gamma)\, t^q \log t\, dt/t - \frac{u^q}{q}, \end{aligned} \tag{4.13}$$

where we have used (4.11) to evaluate the integral involving only t^q. From (4.13) we obtain immediately

$$\frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma)\, t^q \log t\, dt/t = \frac{1}{2} q u^q \log u + \frac{1}{2} u^q \qquad (\gamma = 2q). \tag{4.14}$$

Now (4.14), (4.11), and simple integrations give

$$\begin{aligned} &\frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi+\pi/\gamma}^{\pi-\pi/\gamma} (t^q \log t \cos q\theta - t^q \theta \sin q\theta) \right\} dt/t \\ &\quad = -(-u)^q \log u. \end{aligned} \tag{4.15}$$

Consider finally the case $\psi(z) = \log z$. Then $\exp \psi(z) = z$ and application of Lemma 1 gives

$$\frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi/\gamma}^{\pi/\gamma} \log t\, d\theta \right\} dt/t = \log u, \tag{4.16}$$

from which follows immediately

$$\frac{\gamma^2}{2\pi} \int_0^\infty X(t, u; \gamma) \left\{ \int_{-\pi+\pi/\gamma}^{\pi-\pi/\gamma} \log t\, d\theta \right\} dt/t = (2\lambda - 1) \log u \qquad (\gamma = 2\lambda). \tag{4.17}$$

If $\lambda > \frac{1}{2}$ is nonintegral, we substitute (9) in (4.4) and use (4.11) and (4.14). After some simple reductions we obtain

$$\begin{aligned} \log |P_\lambda(-u)| &= \log |\sin \pi u^\lambda| + (\pi \cot \pi\lambda)\, u^\lambda + \sum_{m=1}^{q} \frac{\zeta(m/\lambda)}{m} u^m \\ &\quad - \tfrac{1}{2} \log u + \log 2 - (1/2\lambda) \log 2\pi - \mathscr{E}_1(u), \end{aligned} \tag{4.18}$$

where $\mathscr{E}_1$ is given by (20).

If $\lambda = q$ is a positive integer, we substitute (17) in (4.4) and use (4.15), (4.11), and (4.16). After some simple reductions we obtain

$$\log|P_\lambda(-u)| = \log|\sin \pi u^\lambda| + u^q \log u + \frac{(c-1)}{q} u^q + \sum_{m=1}^{q-1} \frac{\zeta(m/\lambda)}{m} u^m$$
$$- \tfrac{1}{2} \log u + \log 2 - (1/2\lambda) \log 2\pi - \mathscr{E}_1(u), \tag{4.19}$$

where $\mathscr{E}_1$ is given by (20).

Now if $n^{1/\lambda} < u < (n+1)^{1/\lambda}$, where n is a positive integer, then $P_\lambda(-u)$ and $\sin \pi u^\lambda$ each have the sign of $(-1)^n$. This implies that, after exponentiation, the absolute value signs in (4.18) and (4.19) may be removed. Thus we obtain

$$P_\lambda(-u) = 2(2\pi)^{-1/2\lambda} u^{-1/2} \sin \pi u^\lambda \exp \Big\{ (\pi \cot \pi\lambda)\, u^\lambda$$
$$+ \sum_{m=1}^{q} \frac{\zeta(m/\lambda)}{m} u^m - \mathscr{E}_1(u) \Big\}, \tag{4.20}$$

provided that λ $(>\frac{1}{2})$ is nonintegral.

If $\lambda = q$ is a positive integer the corresponding formula will be

$$P_\lambda(-u) = 2(2\pi)^{-1/2\lambda} u^{-1/2} \sin \pi u^\lambda \exp \Big\{ u^q \log u$$
$$+ \frac{(c-1)}{q} u^q + \sum_{m=1}^{q-1} \frac{\zeta(m/\lambda)}{m} u^m - \mathscr{E}_1(u) \Big\}. \tag{4.21}$$

To finish the proof of Theorem 3, it remains to extend (4.20) and (4.21) to complex values of u; it is clear that this will be possible if we can extend $\mathscr{E}_1(u)$ to an analytic function defined for complex values of u of sufficiently small arguments.

Fix $\lambda > \frac{1}{2}$ and put $\gamma = 2\lambda$. Choose $\delta = \pi/4\gamma$ and consider the region

$$D = D_\delta = \{ z = re^{i\theta} : r > 0, |\theta| \leqslant \delta \}$$

of the complex plane.

For $z \in D$, define $\mathscr{E}_1(z)$ by

$$\mathscr{E}_1(z) = \frac{\gamma^2}{2\pi} \int_0^\infty \frac{t^\gamma z^\gamma}{(t^\gamma + z^\gamma)^2} \left\{ \int_{-\pi+\pi/\gamma}^{\pi-\pi/\gamma} \operatorname{Re} \mathscr{E}(te^{i\theta})\, d\theta \right\} dt/t, \tag{4.22}$$

where $\mathscr{E}(te^{i\theta})$ is defined in (10), and z^γ has its principal value.

Since $z = re^{i\theta} \in D$, it follows that

$$|t^\gamma + z^\gamma| \geqslant \operatorname{Re}(t^\gamma + z^\gamma) = t^\gamma + r^\gamma \cos \gamma\theta \geqslant t^\gamma + r^\gamma \cos \gamma\delta$$
$$\geqslant (t^\gamma + r^\gamma)/\sqrt{2}. \tag{4.23}$$

It is immediately clear, from (9) and (17), that for t near 0,

$$\operatorname{Re} \mathscr{E}(te^{i\theta}) = \tfrac{1}{2} \log t + O(1); \tag{4.24}$$

while for all large t

$$\operatorname{Re} \mathscr{E}(te^{i\theta}) = o(1) \tag{4.25}$$

by the estimates at the end of Section 2.

Using (4.24) and (4.25) in (4.22) we see that, on compact subsets of D, the integral in (4.22) is comparable to the sum of integrals

$$K_1 \int_0^1 t^{\gamma-1} \log t \, dt + K_2 \int_0^1 t^{\gamma-1} \, dt + K_3 \int_1^\infty t^{-\gamma-1} \, dt \qquad (\gamma = 2\lambda > 1),$$

which are convergent and independent of z. It follows that the integral defining $\mathscr{E}_1(z)$ is uniformly convergent on compact subsets of D. Thus $\mathscr{E}_1(z)$ is analytic in D and the proof of formulas (18) and (19) of Theorem 3 is complete.

Once again, use of (4.23) and (4.24) in (4.22) leads to

$$|\mathscr{E}_1(z)| \leqslant K_4 r^{-\gamma} \int_0^1 (\log t + \text{const}) \, t^{\gamma-1} \, dt + K_5(1 + r^\gamma)^{-1}$$

so that $\mathscr{E}_1(z)$ tends to 0 as z tends to ∞ uniformly in D. This finishes the proof of Theorem 3.

5. Applications; Proof of Theorem 4

Assume that λ (>1) is nonintegral. Fix $r > 0$ and let

$$\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_l \tag{5.1}$$

be the points in $(0, \pi)$ where $(\partial/\partial\theta)(\log |P_\lambda(re^{i\theta})|) = 0$. Then [11, p. 500]

$$l = q - 1 \quad \text{or} \quad q, \tag{5.2}$$

and if we set

$$\tilde{\beta}_0 = 0, \qquad \tilde{\beta}_{l+1} = \pi, \tag{5.3}$$

then [11], $\log|P_\lambda(re^{i\theta})|$ is monotonic in each of the intervals

$$[\tilde{\beta}_j, \tilde{\beta}_{j+1}], \qquad j = 0, 1, 2, \ldots, l. \tag{5.4}$$

We first show that, for all large r, the derivative vanishes at exactly q points in $(0, \pi)$, i.e., that $l = q$ in (5.2). Let the arguments φ_k be defined by

$$\varphi_k = k\pi/\lambda, \qquad k = 1, 2, 3, \ldots, q; \tag{5.5}$$

and let

$$\delta = \tfrac{1}{4}\pi(1 - q/\lambda). \tag{5.6}$$

Then $\delta > 0$ and the intervals $|\theta - \varphi_k| \leqslant \delta$ are disjoint and properly contained in $(0, \pi)$. If $|\theta - \varphi_k| \leqslant \delta$, we can use Theorem 1 to compute the second derivative of $\log|P_\lambda(re^{i\theta})|$ with respect to θ and we obtain

$$(\partial^2/\partial\theta^2) \log|P_\lambda(re^{i\theta})| = -\pi\lambda^2 r^\lambda \csc \pi\lambda \cos \lambda\theta + a(r, \theta).$$

The values of θ under consideration are bounded away from π by a fixed positive constant and this allows a uniform estimate of $a(r, \theta)$. We obtain

$$\begin{aligned}(\partial^2/\partial\theta^2) \log|P_\lambda(re^{i\theta})| &= -\pi\lambda^2 r^\lambda \csc \pi\lambda \cos \lambda\theta + O(r^q)\\ &= (-1)^{k+1}\pi\lambda^2 r^\lambda \csc \pi\lambda \cos \lambda(\theta - \varphi_k) + O(r^q),\end{aligned} \tag{5.7}$$

where the constant implied in the O notation depends on λ and q only.

Since $q < \lambda$, the first term eventually dominates and so the sign of the second derivative is determined by the sign of the first term in (5.7) as soon as r exceeds a suitable bound, say r_0. For all θ in $|\theta - \varphi_k| \leqslant \delta$ and all $k = 1, 2, \ldots, q$, we have $\cos \lambda(\theta - \varphi_k) \geqslant \cos \lambda\delta \geqslant \cos \frac{1}{4}\pi > 0$. Thus the sign of the second derivative of $\log|P_\lambda(re^{i\theta})|$ in the interval $|\theta - \varphi_k| \leqslant \delta$ is the same as that of $(-1)^{k+1} \csc \pi\lambda$ for all $r \geqslant r_0$. In particular it is constant in each of these intervals for all $r \geqslant r_0$. It follows that the first derivative of $\log|P_\lambda(re^{i\theta})|$ is monotonic in each of these intervals and so it vanishes at most once in each of them.

Suppose now that $\frac{1}{2}\delta \leqslant |\theta - \varphi_k| \leqslant \delta$. Then, by Theorem 1,

$$\begin{aligned}&\log|P_\lambda(re^{i\varphi_k})| - \log|P_\lambda(re^{i\theta})|\\ &\quad = \pi r^\lambda \csc \pi\lambda(\cos \lambda\varphi_k - \cos \lambda\theta) + O(r^q)\\ &\quad = \pi r^\lambda(-1)^k \csc \pi\lambda \cdot 2 \sin^2 \tfrac{1}{2}\lambda(\theta - \varphi_k) + O(r^q),\end{aligned} \tag{5.8}$$

which, for all large r, has the sign of $(-1)^k$ if q is even and of $(-1)^{k+1}$ if q is odd.

If, for example, q is even and k is even, then $|P_\lambda(re^{i\varphi_k})| > |P_\lambda(re^{i\theta})|$ for all $\frac{1}{2}\delta \leqslant |\theta - \varphi_k| \leqslant \delta$ and all large r; so that in this case the maximum of $|P_\lambda(re^{i\theta})|$ in the interval $|\theta - \varphi_k| \leqslant \delta$ occurs in the smaller interval $\frac{1}{2}\delta \leqslant |\theta - \varphi_k| \leqslant \delta$. At such a point of maximum the derivative will therefore vanish.

If q is even and k is odd, then the minimum occurs in the smaller interval and, again, the derivative will vanish there. A similar situation prevails if q is odd and we conclude that $(\partial/\partial\theta)\log|P_\lambda(re^{i\theta})|$ vanishes exactly once in each of the intervals $|\theta - \varphi_k| \leqslant \delta$ for all large r. Since $|P_\lambda(re^{i\theta})|$ is an even function of θ, the discussion has also shown that, for all large r, $|P_\lambda(re^{i\theta})|$ has exactly $q+1$ points of maximum in $[-\pi, \pi]$, one occurring in each of the intervals

$$|\theta - \theta_k| \leqslant \delta \qquad (k \in J),$$

where J and θ_k are defined by (11) and (12). This finishes the proof of the first part of Theorem 4.

Denote by $\beta_k = \beta_k(r)$ the maximum point of $|P_\lambda(re^{i\theta})|$ occurring in $|\theta - \theta_k| \leqslant \delta$. Then β_k is completely determined by

$$\begin{aligned} (\partial/\partial\theta)\log|P_\lambda(re^{i\theta})| &= 0 \qquad \text{if} \quad \theta = \beta_k; \\ |\beta_k - \theta_k| &\leqslant \tfrac{1}{2}\delta \qquad \text{if} \quad r > r_0. \end{aligned} \tag{5.9}$$

Computing the first derivative from Theorem 1, setting $\theta = \beta_k$, equating to zero, and then dividing by r^λ, we obtain

$$\sin\lambda\beta_k \to 0 \qquad \text{as} \quad r \to \infty, \quad \text{uniformly in } k \in J.$$

Or equivalently,

$$\sin\lambda(\beta_k - \theta_k) \to 0 \qquad \text{as} \quad r \to \infty, \quad \text{uniformly in } k \in J. \tag{5.10}$$

The definition of δ implies that $\lambda\delta < \frac{1}{4}\pi$ so that the second part of (5.9) and an elementary inequality give

$$(2\lambda/\pi)\,|\beta_k - \theta_k| \leqslant \sin\lambda\,|\beta_k - \theta_k| = |\sin\lambda(\beta_k - \theta_k)|.$$

In view of (5.10), this implies that

$$\beta_k \to \theta_k \qquad \text{as} \quad r \to \infty, \quad \text{uniformly in } k \in J. \tag{5.11}$$

This completes the proof of (14).

We next compare the growth of $|P_\lambda(re^{i\theta})|$ on two curves of local maxima. We first show that the constant α_k defined in (13) is positive.

Suppose that k and $k+1$ are nonnegative integers both belonging to the set J defined in (11). If q is even, then

$$2k\pi = (2k+1)\pi - \pi < (2k+1)\pi - \frac{(2k+1)}{q+1}\pi$$
$$= (2k+1)\frac{q\pi}{q+1} < (2k+1)\frac{q\pi}{\lambda} < (2k+1)\pi$$

and this implies that $\sin(2k+1)(q\pi/\lambda) > 0$.

If q is odd, then

$$(2k+1)\pi = (2k+2)\pi - \pi < (2k+2)\frac{q\pi}{q+1} < (2k+2)\frac{q\pi}{\lambda} < (2k+2)\pi,$$

which implies that $\sin(2k+2)(q\pi/\lambda) < 0$.

Recalling the definition of θ_k in (12) and using an elementary trigonometric identity we obtain

$$\cos q\theta_k - \cos q\theta_{k+1} = 2\sin\frac{q\pi}{\lambda}\sin(2k+1)\frac{q\pi}{\lambda} > 0, \qquad \text{if} \quad q \text{ is even,}$$
$$= 2\sin\frac{q\pi}{\lambda}\sin(2k+2)\frac{q\pi}{\lambda} < 0, \qquad \text{if} \quad q \text{ is odd,}$$

since $\sin(q\pi/\lambda) > 0$. It follows that in all cases

$$(-1)^q(\cos q\theta_k - \cos q\theta_{k+1}) > 0 \qquad \text{if} \quad k, k+1 \in J, \quad k \geqslant 0.$$

But $\zeta(q/\lambda) < 0$, because $0 < q/\lambda < 1$, and so, by (13), α_k is positive.

We now prove (15). Retaining the same assumptions on k and $k+1$, we have from Theorem 1 and the definitions of β_{k+1} and θ_{k+1}

$$\begin{aligned}
&\log|P_\lambda(re^{i\beta_k})| - \log|P_\lambda(re^{i\beta_{k+1}})| \\
&\quad < \log|P_\lambda(re^{i\beta_k})| - \log|P_\lambda(re^{i\theta_{k+1}})| \\
&\quad = \pi\csc\pi\lambda r^\lambda\cos\lambda\beta_k - \pi|\csc\pi\lambda|\, r^\lambda \\
&\qquad + ((-1)^q/q)\,\zeta(q/\lambda)\, r^q\{\cos q\beta_k - \cos q\theta_{k+1}\} + O(r^{q-1}). \qquad (5.12)
\end{aligned}$$

The continuity of the cosine, Eq. (5.11), and the finiteness of J imply the existence of an $r_2 > 0$ such that

$$(-1)^q\{\cos q\beta_k - \cos q\theta_{k+1}\} > \tfrac{3}{4}(-1)^q\{\cos q\theta_k - \cos q\theta_{k+1}\} \qquad (5.13)$$

for all $r \geqslant r_2$ and all nonnegative k and $k+1$ in J. Combining (5.13) and (5.12), and recalling (13), we obtain

$$\log |P_\lambda(re^{i\beta_k})| - \log |P_\lambda(re^{i\beta_{k+1}})| < -\tfrac{3}{4}\alpha_k r^q + O(r^{q-1}). \tag{5.14}$$

Since $\alpha_k > 0$ and the O term involves a constant depending on λ and q only, this term will be compensated by $-\frac{1}{4}\alpha_k r^q$ as soon as r exceeds a suitable bound, say r_3. Therefore

$$\log |P_\lambda(re^{i\beta_k})| - \log |P_\lambda(re^{i\beta_{k+1}})| < -\tfrac{1}{2}\alpha_k r^q, \tag{5.15}$$

provided that $r \geqslant r_3$, k is positive, and $k+1 \in J$. This establishes (15).

Now (16) follows immediately from (15) and the fact that $|P_\lambda(re^{i\theta})|$ is an even function of θ in $[-\pi, \pi]$. This completes the proof of Theorem 4.

For the Corollary, if, for example, the coefficients in (2) satisfy $a_n \geqslant 0$ for all $n \geqslant N$, where N is some fixed nonnegative integer, then it is easy to conclude that $|P_\lambda(z)| \leqslant P_\lambda(r) + O(r^N)$, from which it follows immediately that $M(r; P_\lambda) \sim P_\lambda(r)$ as $r \to \infty$. But this last asymptotic equality contradicts (15). Hence the coefficients a_n in (2) cannot maintain a constant sign. This proves Corollary 1.

6. Applications; Proof of Theorem 5

Let

$$P_\lambda(z) = \sum_{n=0}^{\infty} a_n z^n \tag{6.1}$$

be the Taylor's series expansion of P_λ about the origin. In this section we study the behavior of the coefficients a_n for large values of n. Such a study we carry out by expressing a_n as an integral by Cauchy's formula and then using a saddle point technique [5] to study the integral. The method requires

(a) Determination of the points on the circle $|z| = r$ where $|P_\lambda(z)|$ assumes a local maximum.

(b) Precise assessment of the contribution to the Cauchy integral from immediate neighborhoods of such points.

(c) Estimation of the contributions to the integral from the remaining arcs on $|z| = r$.

Step (a) has been completed in Theorem 4. Steps (b) and (c) will be taken up in this section. For step (b), it will turn out that the main contribution comes only from the immediate vicinities of the two points in $(-\pi, \pi)$ where the maximum modulus of $P_\lambda(z)$ occurs. Step (c) will be completed easily from Theorem 4 and the monotonic behavior of $|P_\lambda(re^{i\theta})|$ in $[0, \pi]$.

Throughout this section we retain the notations of Theorem 4; thus J, θ_k, and α_k are the symbols defined in (11)–(13). The points of local maximum of $|P_\lambda(re^{i\theta})|$ in $[-\pi, \pi]$ will be denoted, as before, by β_k, where $k \in J$. To avoid a separation of the discussion into two cases, we let $\omega = \frac{1}{2}q$ if q is even and $\frac{1}{2}(q-1)$ if q is odd. Then $\beta_\omega = \beta_\omega(r)$ will always mean the point in $(0, \pi)$ where the maximum modulus of $P_\lambda(z)$ occurs. Furthermore, for each fixed sufficiently large r, β_ω is uniquely determined by the conditions

$$\begin{aligned} &(\partial/\partial\theta) \log |P_\lambda(re^{i\theta})| = 0 && \text{at} \quad \theta = \beta_\omega; \\ &|\beta_\omega - \theta_\omega| = |\beta_\omega - (q\pi/\lambda)| \leqslant \delta = \tfrac{1}{4}\pi(1 - q/\lambda) && (r \geqslant r_0). \end{aligned} \tag{6.2}$$

If we compute the derivative indicated in (6.2) by use of Theorem 1, the equation in (6.2) takes the form

$$\begin{aligned} -\pi\lambda r^\lambda \csc \pi\lambda \sin \lambda\beta_\omega - \sum_{m=1}^{q} (-1)^m \zeta(m/\lambda)\, r^m \sin m\beta_\omega \\ + (\partial/\partial\theta) \operatorname{Re} \mathscr{E}(re^{i\beta_\omega}) = 0. \end{aligned} \tag{6.3}$$

The derivative of $\operatorname{Re} \mathscr{E}(re^{i\theta})$ may be estimated in the same manner as $\mathscr{E}(re^{i\theta})$ and we obtain from (6.3)

$$|r^\lambda \sin \lambda\beta_\omega| \leqslant K_1 r^q \qquad (r \geqslant r_1 > 0), \tag{6.4}$$

where r_1 and K_1 are constants depending on q and λ only.

In the course of our proof we shall use K_1, K_2,... to denote constants depending on q and λ only. The notation $(r \geqslant r_0)$ following an inequality means that the inequality is valid for sufficiently large values of r; the bound r_0 may depend on q and λ. The dependence of r or K on other parameters, such as ε, θ, and r will be indicated by writing, e.g., $r(\varepsilon)$, $K(r, \theta)$, etc.

In order to carry out steps (b) and (c) indicated at the beginning of this section we introduce two functions $a(r)$ and $b(r)$. They are defined for positive r as follows:

$$\begin{aligned} a(r) &= \pi\lambda r^\lambda \csc \pi\lambda \cos \lambda\beta_\omega + \sum_{m=1}^{q} (-1)^m \zeta(m/\lambda)\, r^m \cos m\beta_\omega - \tfrac{1}{2}, \\ b(r) &= \pi\lambda^2 r^\lambda \csc \pi\lambda \cos \lambda\beta_\omega + \sum_{m=1}^{q} m(-1)^m \zeta(m/\lambda)\, r^m \cos m\beta_\omega. \end{aligned} \tag{6.5}$$

By (14), $\beta_\omega \to \theta_\omega = q\pi/\lambda$. Thus $\csc \pi\lambda \cos \lambda\beta_\omega > 0$ for all large r; and since $q < \lambda$, the first terms in the definitions of $a(r)$ and $b(r)$ eventually dominate so that $a(r)$ and $b(r)$ will be positive for all large r and

$$a(r) \to \infty \qquad \text{and} \qquad b(r) \to \infty \qquad \text{as} \quad r \to \infty. \tag{6.6}$$

The Taylor's series expansions of the sine and cosine functions about β_ω give

$$e^{ix\theta}=e^{ix\beta_\omega}+ix(\theta-\beta_\omega)\,e^{ix\beta_\omega}-\tfrac{1}{2}x^2(\theta-\beta_\omega)^2\,e^{ix\beta_\omega}$$
$$+(\theta-\beta_\omega)^3\,K(\theta;\beta_\omega), \tag{6.7}$$

where $|K(\theta;\beta_\omega)|\leqslant x^3$. Using (6.7) once with $x=\lambda$ and another time with $x=m$ (where $m=1,2,\ldots,q$) and Theorem 1 and then (6.5), we obtain

$$\begin{aligned}
&\log P_\lambda(re^{i\theta})-\log P_\lambda(re^{i\beta_\omega})\\
&=\pi r^\lambda\csc\pi\lambda(e^{i\lambda\theta}-e^{i\lambda\beta_\omega})+\sum_{m=1}^{q}\frac{(-1)^m}{m}\zeta(m/\lambda)\,r^m(e^{im\theta}-e^{im\beta_\omega})\\
&\quad-\tfrac{1}{2}i(\theta-\beta_\omega)+\mathscr{E}(re^{i\theta})-\mathscr{E}(re^{i\beta_\omega})\\
&=i(\theta-\beta_\omega)\,a(r)-\tfrac{1}{2}(\theta-\beta_\omega)^2\,b(r)+\mathscr{E}(re^{i\theta})-\mathscr{E}(re^{i\beta_\omega})\\
&\quad-(\theta-\beta_\omega)\left\{\pi\lambda r^\lambda\csc\pi\lambda\sin\lambda\beta_\omega+\sum_{m=1}^{q}(-1)^m\,\zeta(m/\lambda)\,r^m\sin m\beta_\omega\right\}\\
&\quad-\tfrac{1}{2}i(\theta-\beta_\omega)^2\left\{\pi\lambda^2r^\lambda\csc\pi\lambda\sin\lambda\beta_\omega+\sum_{m=1}^{q}m(-1)^m\,\zeta(m/\lambda)\,r^m\sin m\beta_\omega\right\}\\
&\quad+(\theta-\beta_\omega)^3\,K_3(r;\omega).
\end{aligned}\tag{6.8}$$

In the fifth term, the coefficient of $(\theta-\beta_\omega)$ is, by (6.3), equal to $-(\partial/\partial\theta)\operatorname{Re}\mathscr{E}(re^{i\theta})$ evaluated at $\theta=\beta_\omega$. Estimates of this derivative similar to those of $\mathscr{E}(re^{i\theta})$ are easily obtained, and we see that it tends to 0 as r tends to ∞. In the sixth term, the coefficient of $i(\theta-\beta_\omega)^2$ is, by (6.4), of the order of r^q as $r\to\infty$. It follows that we can write

$$\begin{aligned}
&\log P_\lambda(re^{i\theta})-\log P_\lambda(re^{i\beta_\omega})\\
&\quad=i(\theta-\beta_\omega)\,a(r)-\tfrac{1}{2}(\theta-\beta_\omega)^2\,b(r)+\mathscr{E}(re^{i\theta})-\mathscr{E}(re^{i\beta_\omega})\\
&\qquad-(\theta-\beta_\omega)\,K_1(r;\omega)-\tfrac{1}{2}i(\theta-\beta_\omega)^2\,K_2(r;\omega)+(\theta-\beta_\omega)^3\,K_3(r;\omega),
\end{aligned}\tag{6.9}$$

where

$$|K_2(r;\omega)|\leqslant K_2r^q,\qquad |K_3(r;\omega)|\leqslant K_3r^\lambda\qquad (r\geqslant r_1) \tag{6.10}$$

and

$$K_1(r;\omega)\to 0\qquad\text{as}\quad r\to\infty.$$

In order to get a manageable expression from (6.9), we have to restrict θ to smaller and smaller neighborhoods of β_ω. That is, we have to introduce a function $\delta=\delta(r)$ such that, when $|\theta-\beta_\omega|\leqslant\delta$, all terms on the right-hand side of (6.9), save the first two, tend to 0 as $r \to \infty$ uniformly in θ. In view of the previous discussion, and in particular (6.10), it is clear that such a function δ must satisfy

$$\delta^3 b(r) \to 0, \qquad \delta^2 r^q \to 0 \qquad \text{as} \quad r \to \infty. \tag{6.11}$$

For later use, it will also be necessary to require that the function δ satisfy the additional condition

$$\delta^2 b(r) \to \infty \qquad \text{as} \quad r \to \infty. \tag{6.12}$$

We now define the function $\delta = \delta(r)$. Recalling that $q = [\lambda]$ and that $q \geqslant 1$, let

$$\alpha = (q + \lambda)/8q, \qquad \beta = (q + \lambda)/8\lambda,$$

and define δ by

$$\delta = \delta(r) = r^{-\alpha q - \beta\lambda} = r^{-(1/4)(q+\lambda)}. \tag{6.13}$$

If $q \geqslant 2$, then $\frac{1}{3}\lambda < \frac{1}{2}q < \alpha q + \beta\lambda < \frac{1}{2}\lambda$; while if $q = 1$, then $\frac{1}{2}q < \alpha q + \beta\lambda < \frac{1}{2}\lambda$ and $\alpha q + \beta\lambda > \frac{1}{3}\lambda$. Since $b(r) \sim r^{\lambda}$, these inequalities imply that both (6.11) and (6.12) are satisfied by the function defined in (6.13).

Returning to (6.9) and using (6.10) and (6.11), we conclude that

$$P_\lambda(re^{i\theta}) \sim P_\lambda(re^{i\beta_\omega}) \exp\{i(\theta - \beta_\omega)\, a(r) - \tfrac{1}{2}(\theta - \beta_\omega)^2\, b(r) \tag{6.14}$$

as r tends to ∞ unifiormly in $|\theta - \beta_\omega| \leqslant \delta(r)$.

This completes step (b) mentioned in the introduction. We now estimate $|P_\lambda(re^{i\theta})|$ in the remaining parts of $[-\pi, \pi]$. Suppose that $\theta \in [0, \pi]$ and satisfies $|\theta - \beta_\omega| \geqslant \delta$. Then the monotonic behavior [11, p. 500] of $|P_\lambda(re^{i\theta})|$ implies that

$$|P_\lambda(re^{i\theta})| \leqslant \max_k\, (|P_\lambda(re^{i(\beta_\omega \pm \delta)}); |P_\lambda(re^{i\beta_k})|), \tag{6.15}$$

where $k \in J$ and $0 \leqslant k \leqslant \omega$, and for all $r \geqslant r_1$.

Write $\psi = \beta_\omega \pm \delta$. In (6.9), put $\theta = \psi$ and take real parts. In view of (6.11), the result is

$$|P_\lambda(re^{i\psi})|/|P_\lambda(re^{i\beta_\omega})| = O(\exp\{-\tfrac{1}{2}\delta^2 b(r)\}) \qquad (r \to \infty). \tag{6.16}$$

Now (6.15), (6.16), repeated application of (15), and (6.12) give

$$\sqrt{b(r)}\, |P_\lambda(re^{i\theta})| = o(|P_\lambda(re^{i\beta_\omega})|) \tag{6.17}$$

as $r \to \infty$ uniformly in θ in $|\theta - \beta_\omega| \geqslant \delta = \delta(r)$.

This completes step (c) above and we now turn to the evaluation of the Cauchy integral that represents a_n.

By Cauchy's formula and the equality $P_\lambda(re^{i\theta}) = \overline{P_\lambda(re^{i\theta})}$ we have

$$a_n r^n = \operatorname{Re}\left(\frac{1}{\pi}\int_0^\pi P_\lambda(re^{i\theta})\, e^{-in\theta}\, d\theta\right)$$

$$= \operatorname{Re}\frac{1}{\pi}\left\{\int_A + \int_B P_\lambda(re^{i\theta})\, e^{-in\theta}\, d\theta\right\}, \tag{6.18}$$

where

$$A = \{\theta \in [0, \pi] : |\theta - \beta_\omega| \leqslant \delta\}, \qquad B = [0, \pi] - A.$$

By (6.17), the integral over B is $o(|P_\lambda(re^{i\beta_\omega})|/\sqrt{b(r)})$.

By (6.14) and a change of variable, the integral over A equals

$$P_\lambda(re^{i\beta_\omega})\int_{-\delta}^{\delta} \{1 + o(1)\} \exp\{i\theta a(r) - \tfrac{1}{2}\theta^2 b(r)\}\, e^{-in(\theta+\beta_\omega)}\, d\theta, \tag{6.19}$$

which, except for the factor $e^{-in\beta_\omega}$, is precisely the integral obtained in [5, p. 71] for admissible functions. As in [5], the change of variable $y = \theta\sqrt{(1/2)\, b(r)}$ gives

$$\sqrt{(2/b(r))}\int_{-\delta(r)\sqrt{(1/2)b(r)}}^{\delta(r)\sqrt{(1/2)b(r)}} \exp[i\{a(r) - n\}\, y\sqrt{(2/b(r))} - y^2]\, dy \tag{6.20}$$

for the dominant term in the integral in (6.19).

Since $\delta^2 b(r) \to \infty$, by (6.12) the integral in (6.20) may be extended over the whole interval $(-\infty, \infty)$ at the expense of a small error term. The resulting integral over $(-\infty, \infty)$ may be evaluated by turning the line of integration in the obvious way and we find that it equals

$$\sqrt{\pi}\, \exp\{-\tfrac{1}{2}(a(r) - n)^2/b(r)\}.$$

An easy estimate of the error term in (6.19), together with the above calculations gives

$$a_n r^n = \operatorname{Re}\sqrt{\frac{2}{\pi b(r)}}\, P_\lambda(re^{i\beta_\omega})\, e^{-in\beta_\omega}\left\{\exp\left[-\frac{(a(r) - n)^2}{2b(r)}\right] + o(1)\right\} \tag{6.21}$$

as $r \to \infty$, uniformly for all $n \geqslant 0$.

To obtain the best information from (6.21) we have to choose $r = r(n)$ in such a way that $a(r) = n + o(\sqrt{b(r)})$ as $n \to \infty$. Such a choice will be possible if, e.g., the equation $a(r) = n$ can be solved for r as soon as n

becomes sufficiently large. The definitions of $a(r)$ and $b(r)$ and Eq. (14) show that, if r_n is a solution of the equation $a(r) = n$, then

$$b(r_n) \sim \lambda a(r_n) = \lambda n, \qquad r_n \sim \{(n\,|\sin \pi\lambda|)/\pi\lambda\}^{1/\lambda} \qquad (n \to \infty).$$

If this information is used in (6.21), we arrive at the asymptotic formula

$$a_n = \sqrt{\frac{2}{\pi\lambda n}} \left\{\frac{n\,|\sin \pi\lambda|}{\pi\lambda}\right\}^{-(n+1/2)\lambda} \times (2\pi)^{-(1/2\lambda)}\, e^{n/\lambda + O(n^{q/\lambda})}\{\cos H(n) + o(1)\}, \tag{6.22}$$

where

$$\begin{aligned} H(r) &= \pi r^\lambda \csc \pi\lambda \sin \lambda\beta_\omega + \sum_{m=1}^{q} \frac{(-1)^m}{m} \zeta(m/\lambda)\, r^m \sin m\beta_\omega \\ &\quad - (n + \tfrac{1}{2})\,\beta_\omega + \operatorname{Im} \mathscr{E}(re^{i\theta}), \qquad (6.23)\\ H(n) &= H(r_n), \end{aligned}$$

and r_n is defined, for all large n, by $a(r_n) = n$.

In order to justify the above computations, we have to show that, for all large r, the equation $a(r) = n$ has a solution $r = r_n$ tending to ∞ as n tends to ∞. This we do by showing that $a(r)$ is eventually a strictly increasing function of r. We need to compute $da(r)/dr$, and this requires, first, the computation of $d\beta_\omega(r)/dr$.

Let $G(r, \theta)$ be defined by

$$\begin{aligned} G(r, \theta) &= \pi\lambda r^\lambda \csc \pi\lambda \sin \lambda\theta + \sum_{m=1}^{q} (-1)^m \zeta(m/\lambda)\, r^m \sin m\theta \\ &\quad + (\partial/\partial\theta) \operatorname{Re} \mathscr{E}(re^{i\theta}) \\ &= (\partial/\partial\theta) \log |P_\lambda(re^{i\theta})|. \end{aligned} \tag{6.24}$$

By (6.2), $\beta_\omega = \beta_\omega(r)$ is defined as the unique solution of the problem

$$\begin{aligned} &G(r, \theta) = 0, \\ &|\theta - \theta_\omega| = |\theta - q\pi/\lambda| \leqslant \delta = \tfrac{1}{4}\pi(1 - q/\lambda) \qquad (r \geqslant r_0). \end{aligned} \tag{6.25}$$

Implicit differentiation of (6.25) and straightforward estimates give

$$d\beta_\omega(r)/dr = -r^{-1} \tan \lambda\beta_\omega + O(r^{q-\lambda-1}) + O(r^{2q-2\lambda-1}). \tag{6.26}$$

Let $A(r, \theta)$ be the function defined by

$$A(r, \theta) = \pi\lambda r^\lambda \csc \pi\lambda \cos \lambda\theta + \sum_{m=1}^{q} (-1)^m \zeta(m/\lambda)\, r^m \cos m\theta - \tfrac{1}{2}$$

Then $a(r) = A(r, \beta_\omega(r))$, and

$$d\,a(r)/dr = A_r + A_\theta\, d\theta/dr \qquad (\theta = \beta_\omega(r)).$$

It follows from this and (6.26), that

$$da(r)/dr = \pi\lambda^2 r^{\lambda-1} \csc \pi\lambda \sec \lambda\theta + O(r^{q-1}) + O(r^{2q-2\lambda+\lambda-1}), \tag{6.27}$$

where $\theta = \beta_\omega(r)$.

Since $q < \lambda$, $2q - 2\lambda + \lambda - 1 < \lambda - 1$ and the first term in (6.27) dominates. Furthermore, $\csc \pi\lambda \sec \lambda\theta$ is positive for all θ in $|\theta - \theta_\omega| \leqslant \delta$. Thus $da(r)/dr$ is positive for all large r and $a(r)$ is an increasing function there. In particular, the equation $a(r) = n$ has a unique root r_n for all large values of n and (6.22) and (6.23) are true for all large values of n. It also follows that $r_n \to \infty$ as $n \to \infty$. This finishes the proof of part I of Theorem 5.

It remains for us to study the sign of $\cos H(n) = \cos H(r_n)$ for all large n.

Consider the system

$$G(r, \theta) = 0, \qquad A(r, \theta) = n, \tag{6.28}$$

where G and A are the functions defined in the previous discussion and n is a *continuous* real variable.

The preceding discussion has shown that there exists n_0 such that, for each $n \geqslant n_0$, system (6.28) has a unique solution

$$r = r(n), \qquad \theta = \theta(n)$$

if $|\theta - \theta_\omega| \leqslant \delta$. Thus system (6.28) defines two functions of n implicitly. If they exist, the derivatives of these functions with respect to n will be given by the formula

$$dr/dn = -G_\theta/J, \qquad d\theta/dn = G_r/J, \qquad J = G_r A_\theta - A_r G_\theta, \tag{6.29}$$

provided that $J \neq 0$.

Straightforward differentiation followed by easy estimates gives

$$\begin{aligned} G_r &= \pi\lambda^2 r^{\lambda-1} \csc \pi\lambda \sin \lambda\theta + O(r^{q-1}), \\ G_\theta &= \pi\lambda^2 r^{\lambda} \csc \pi\lambda \cos \lambda\theta + O(r^{q}), \\ A_r &= \pi\lambda^2 r^{\lambda-1} \csc \pi\lambda \cos \lambda\theta + O(r^{q-1}), \\ A_\theta &= -\pi\lambda^2 r^{\lambda} \csc \pi\lambda \sin \lambda\theta + O(r^{q}), \end{aligned} \tag{6.30}$$

so that

$$J = -(\pi\lambda^2 \csc \pi\lambda)^2\, r^{2\lambda-1} + O(r^{q+\lambda-1}) + O(r^{2q-1}). \tag{6.31}$$

Thus indeed J does not vanish for all large r (and so for all large n) and the functions defined by system (6.28) are differentiable with respect to n and their derivatives are given by (6.29). Using (6.30) and (6.31) in (6.29) we obtain after some easy estimations

$$
\begin{aligned}
dr/dn &= \frac{\cos\lambda\theta}{(\pi\lambda^2 \csc\pi\lambda)\, r^{\lambda-1}} + O(r^{q+1-2\lambda}) \qquad (r = r(n), \theta = \theta(n)),\\
d\theta/dn &= \frac{-\sin\lambda\theta}{(\pi\lambda^2 \csc\pi\lambda)\, r^{\lambda}} + O(r^{q-2\lambda}).
\end{aligned}
\tag{6.32}
$$

We now turn to the function $H(n)$ defined in (6.23). We first show that H is a decreasing function for all large values of n. We start by eliminating the term in $H(n)$ containing r_n^{λ}. Solving the first equation in (6.28) for r^{λ} and using the answer in the definition of $H(n)$ we obtain

$$
\begin{aligned}
H(n) &= \sum_{m=1}^{q} (-1)^m \left(\frac{1}{m} - \frac{1}{\lambda}\right) \zeta(m/\lambda)\, r^m \sin m\theta \\
&\quad - (n + \tfrac{1}{2})\theta + \operatorname{Im} \mathscr{E}(re^{i\theta}) + (1/\lambda) \operatorname{Re} \mathscr{E}_\theta(re^{i\theta}) \\
&\equiv h(r, \theta, n),
\end{aligned}
\tag{6.33}
$$

where $\theta = \theta(n)$ and $r = r(n)$ are defined by (6.28).

The derivative of H with respect to n is given by the formula

$$dH/dn = h_r \cdot dr/dn + h_\theta \cdot d\theta/dn + h_n,$$

and use of (6.32) and easy estimates give

$$
\begin{aligned}
dH/dn &= \frac{q(-1)^q}{\pi\lambda^2 \csc\pi\lambda} \left(\frac{1}{q} - \frac{1}{\lambda}\right) \zeta(q/\lambda)\, r^{q-\lambda} \sin(q-\lambda)\theta \\
&\quad + (n + \tfrac{1}{2}) \frac{\sin\lambda\theta}{(\pi\lambda^2 \csc\pi\lambda)\, r^{\lambda}} - \theta + O(r^{q-\lambda-1}) + O(r^{2q-2\lambda}).
\end{aligned}
\tag{6.34}
$$

As $n \to \infty$, all terms in (6.34) tend to zero except the term θ which tends to $\theta_\omega = q\pi/\lambda$. This implies that $dH/dn < 0$ for all large values of n. It follows that $H(n)$ is a strictly decreasing unbounded function of n. Together with the continuity, this implies that H assumes exactly once every large negative value. Following Edrei [2, p. 224], we define two strictly decreasing unbounded sequences

$$\nu_0, \nu_1, \nu_2, \nu_3, \ldots, \qquad \mu_0, \mu_1, \mu_2, \mu_3, \ldots$$

such that

$$H(\nu_k) = -\pi(l + k + \tfrac{1}{2}) - \eta, \qquad (0 < \eta < \pi/2)$$
$$H(\mu_k) = -\pi(l + k + \tfrac{3}{2}) + \eta \qquad (k = 0, 1, 2, 3,\ldots). \tag{6.35}$$

Taking l to be a large even integer, we conclude that

$$\operatorname{sign}(\cos(H(n))) = (-1)^{k+1} \qquad \text{and} \qquad |\cos H(n)| \geqslant \sin \eta > 0 \tag{6.36}$$

for

$$\nu_k \leqslant n \leqslant \mu_k. \tag{6.37}$$

Formulas (6.25), (6.36), and (6.37) imply that the coefficients a_n maintain a constant sign in each of the intervals $[\nu_k, \mu_k]$, the sign changing as we pass from one such interval to the next. This finishes our discussion of the sign of a_n and we now turn to the last part of this study, namely the asymptotic behavior of the length of the intervals $[\nu_k, \mu_k]$ and $[\mu_k, \nu_{k+1}]$.

Applying the mean value theorem to H on the interval $[\nu_k, \mu_k]$ and taking into account (6.35) and (6.34), we obtain

$$2\eta - \pi = H(\mu_k) - H(\nu_k) = (\mu_k - \nu_k)(-\theta_\omega + o(1)),$$

from which it follows that

$$\mu_k - \nu_k \to (\lambda/q)(1 - (2\eta/\pi)) \qquad \text{as} \quad k \to \infty. \tag{6.38}$$

Similarly, we obtain

$$\nu_{k+1} - \mu_k \to (\lambda/q)(2\eta/\pi). \tag{6.39}$$

Now (6.38) and (6.39) imply

$$\nu_{k+1} - \nu_k \to \lambda/q \tag{6.40}$$

from which it follows immediately that

$$\nu_k \sim (\lambda/q)k. \tag{6.41}$$

Now (6.41) and (6.38) imply

$$\mu_k \sim (\lambda/q)k. \tag{6.42}$$

This completes the proof of Theorem 5.

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